Continuous control of chaos based on the stability criterion

Hong Jie Yu,* Yan Zhu Liu,† and Jian Hua Peng‡
Department of Mechanics, Shanghai Jiao Tong University, 200240 Shanghai, China

(Received 24 June 2003; revised manuscript received 17 December 2003; published 2 June 2004)

A method of chaos control based on stability criterion is proposed in the present paper. This method can stabilize chaotic systems onto a desired periodic orbit by a small time-continuous perturbation nonlinear feedback. This method does not require linearization of the system around the stabilized orbit and only an approximate location of the desired periodic orbit is required which can be automatically detected in the control process. The control can be started at any moment by choosing appropriate perturbation restriction condition. It seems that more flexibility and convenience are the main advantages of this method. The discussions on control of attitude motion of a spacecraft, Rössler system, and two coupled Duffing oscillators are given as numerical examples.

DOI: 10.1103/PhysRevE.69.066203 PACS number(s): 05.45.Gg, 05.45.Pq, 05.45.Xt, 05.70.−a

I. INTRODUCTION

In recent years, the chaotic control and the chaotic synchronization have been widely studied [1–16]. Controlling chaos is a very attractive subject initiated by Ott, Grebogi, and Yorke (OGY) [1]. They proposed an efficient method (OGY method) of chaos control. This method has the ability to stabilize a desired orbit chosen from the many unstable orbits embedded within a chaotic attractor. This aim is achieved by making a small time-dependent perturbation in the form of feedback to accessible system parameter. In recent years its usefulness has been shown by application to many practical systems [4–7]. Some extensions of the OGY method have been proposed [8,9]. Besides, a number of different methods have been developed and applied using the related concepts [10–16].

A nonlinear system with chaotic behavior is very sensitive to initial conditions, particularly in the system with large Lyapunov exponents [17], that a tiny error may lead to failure of the control process when its errors are amplified exponentially with time. Such errors can be introduced by the linearization of a nonlinear system, the inaccuracy of experimental measurement, and the noisy environment. A number of presented methods modify control parameters once each period of Poincaré map [1,8,15,18], and the stabilization can be realized only for such periodic orbits whose maximal Lyapunov exponent is smaller than the reciprocal of the time interval between parameter changes. For the control system with large Lyapunov exponent or high-order unstable periodic orbits, the tiny errors may "kick" the system state out of its controllable region. The fluctuation noise leads to occasional bursts of the system into the region far from the desired periodic orbit, and these bursts are more frequent for a large noise. Therefore the idea of adjusting the system state more frequently than once each period T [15,19], and the idea of a time-continuous control seem attractive in this context [10].

Pyragas have proposed two methods of permanent chaos control with a small time-continuous perturbation in the form of linear feedback [10]. The stabilization of unstable periodic orbits (UPOs) of a chaotic system is achieved either by combined linear feedback with the use of a specially designed external oscillator or by delayed self-controlling linear feedback without any external force. They have calculated the maximal Lyapunov exponent of the UPOs using the linearization of system to analyze the local stability of the system and to select suitable experimentally adjustable weight parameter K. Both methods are based on the construction of a special form of a time-continuous perturbation, which does not change the desired UPO, but can stabilize it under certain conditions.

Ushio proposed a method of chaos control for stabilizing a periodic orbit embedded in a discrete-time chaotic system based on contraction mappings in 1995 [20]. The validity of the method is shown using a property of contraction mappings.

An open-plus-closed-loop (OPCL) method of controlling nonlinear dynamic systems was presented by Atlee Jackson and Grosu in 1995 [12]. The input signal of their method is the sum of Hübler's open-loop control and a particular form of a linear closed-loop control, the goal of which can be selected as one of the UPOs embedded in chaotic attractor, or another possible smooth functions of t. The asymptotic stability of the controlled nonlinear system is realized by the linear approximation around the stabilized orbit. But the calculation of the closed-loop control signal is very difficult in some cases, especially for complex and high-dimension chaotic systems.

In this paper, inspired by the continuous linear feedback control method [10] and the contraction mapping control method of discrete system [20], as well as the OPCL method [12], we propose a method for controlling chaos in the form of special nonlinear feedback. The validity of this method is based on the stability criterion of linear system, and it can be called stability criterion method (SC method). The construction of a nonlinear form of a time-continuous perturbation feedback by a suitable separation of the systems in the SC method does not change the form of the desired UPO. The close return pair technique [5] is utilized to estimate a de-

*Corresponding author. Email address: yuhongjie@sjtu.edu.cn
†Email address: iuyzhc@online.sh.cn
‡Email address: jhpeng@sjtu.edu.cn

sired periodic orbit chosen from numerous UPOs embedded within a chaotic attractor.

Using the SC method, the effect of the control can be guaranteed directly without calculation of the maximal Lyapunov exponent of the UPOs using the linearization of system as in Ref. [10]. This method does not require linearization of the system around the stabilized orbit and calculation of the derivative at UPOs. It seems simpler than the OPCL method and OGY method in controlling UPOs. As examples of numerical simulation, the control of the Rössler system, the control of chaotic attitude motion of a spacecraft, and the control of two coupled Duffing oscillators are investigated.

II. THE STABILITY CRITERION METHOD

We consider a time-continuous nonlinear dynamic system with input perturbation described by

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{u}(t), \tag{1}$$

where $\boldsymbol{x} \in \mathbf{R}^n$ and $\boldsymbol{u} \in \mathbf{R}^n$ are the state vector and input perturbation of the system, respectively. Equation (1) without input signal $(u=0)$ has a chaotic attractor Ω. A mapping $\boldsymbol{f}: \mathbf{R}^n \to \mathbf{R}^n$ is defined in n-dimensional space. We suitably decompose the function $\boldsymbol{f}(\boldsymbol{x}(t))$ as

$$\boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{g}(\boldsymbol{x}(t)) + \boldsymbol{h}(\boldsymbol{x}(t)), \tag{2}$$

where the function $\boldsymbol{g}(\boldsymbol{x}(t)) = \mathbf{A}\boldsymbol{x}(t)$ is suitably disposed as a linear part of $\boldsymbol{f}(\boldsymbol{x}(t), t)$, and it is required that $\mathbf{A}$ is a full rank constant matrix, all eigenvalues of which have negative real parts. So the function $\boldsymbol{h}(\boldsymbol{x}(t)) = \boldsymbol{f}(\boldsymbol{x}(t)) - \mathbf{A}\boldsymbol{x}$ is a nonlinear part of $\boldsymbol{f}(\boldsymbol{x}(t))$. Then the system (1) can be rewritten as

$$\frac{d\boldsymbol{x}}{dt} = \mathbf{A}\boldsymbol{x}(t) + \boldsymbol{h}(\boldsymbol{x}(t)) + \boldsymbol{u}(t). \tag{3}$$

Let $\boldsymbol{D}(\boldsymbol{x}(t)) = -\boldsymbol{h}(\boldsymbol{x}(t))$, we can see that the function $\boldsymbol{f}+\boldsymbol{D}=\boldsymbol{f}-\boldsymbol{h}$ is a linear mapping with respect to the state vector $\boldsymbol{x}$, namely,

$$(\boldsymbol{f} + \boldsymbol{D})(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}. \tag{4}$$

Let $\boldsymbol{x}^*(t) = \boldsymbol{x}^*(t + jT)$, $j = 1, 2, \ldots$, be a period-j trajectory embedded within Ω. The input signal $\boldsymbol{u}(t)$ is considered as a control perturbation signal as follows:

$$\boldsymbol{u}(t) = \boldsymbol{D}(\boldsymbol{x}(t)) - \boldsymbol{D}(\boldsymbol{x}^*(t)) \tag{5}$$

Substituting Eq. (5) into Eq. (3), system (1) and (3) can be rewritten as

$$\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}^* = (\boldsymbol{f} + \boldsymbol{D})(\boldsymbol{x}) - (\boldsymbol{f} + \boldsymbol{D})(\boldsymbol{x}^*) = \mathbf{A}(\boldsymbol{x} - \boldsymbol{x}^*) \tag{6}$$

The difference between $\boldsymbol{x}(t)$ and $\boldsymbol{x}^*(t)$ is defined as an error $\boldsymbol{w}(t) = \boldsymbol{x}(t) - \boldsymbol{x}(t)^*$, the evolution of which is determined by Eq. (6) as

$$\dot{\boldsymbol{w}}(t) = \mathbf{A}\boldsymbol{w}(t). \tag{7}$$

Obviously, the zero point of $\boldsymbol{w}(t)$ is its equilibrium point. Since all eigenvalues of the matrix $\mathbf{A}$ have negative real parts, according to the stability criterion of linear system, the zero point of error $\boldsymbol{w}(t)$ is asymptotically stable and $\boldsymbol{w}(t)$ tends to zero when $t \to \infty$. Then the state vector $\boldsymbol{x}(t)$ tends to the period-j trajectory $\boldsymbol{x}^*(t)$. It implies that the unstable periodic orbit is stabilized. Note that the input perturbation $\boldsymbol{u}(t)$ becomes zero after the state of the controlled system converges to the UPO.

Some very complicated periodically driven dynamic systems along with the stabilized UPO can have alternative stable solutions belonging to different basins of initial conditions. Besides, large initial values of the perturbation can be also undesired for some experiments. Such problems can be solved by restriction of the perturbation. Therefore the stabilization is achieved by small input values when Eq. (5) is modified as follows:

$$\begin{aligned}\boldsymbol{u}(t) &= \boldsymbol{D}(\boldsymbol{x}(t)) - \boldsymbol{D}(\boldsymbol{x}^*(t)) & \\ &= \mathbf{A}(\boldsymbol{x} - \boldsymbol{x}^*) + \boldsymbol{f}(\boldsymbol{x}^*) - \boldsymbol{f}(\boldsymbol{x}) \qquad & \text{if } |\boldsymbol{x} - \boldsymbol{x}^*| < \varepsilon, \\ &= 0 & \text{otherwise}\end{aligned} \tag{8}$$

where $\varepsilon(\varepsilon > 0)$ is a restriction value of error within which $\boldsymbol{u} \neq 0$. The perturbation $\boldsymbol{u}(t)$ is treated as a nonlinear feedback form. In fact, when the condition (2) is satisfied by a suitable separation of system (1), UPOs can be stabilized based on stability criterion of error linear system (7). Moreover, the perturbation $\boldsymbol{u}(t)$ has a simple form as shown in Eq. (8). It is not needed to calculate the derivative $d\boldsymbol{f}/d\boldsymbol{x}$ at the UPO as required in the OPCL control method [12].

In order to obtain the necessary information on an appropriate location of a desired periodic solution $\boldsymbol{x}^*$, the strategy of the close return pairs described in Refs. [8,15] is utilized. A time series of the chaotic trajectory generated by the system (1) is stroboscopically sampled in every period T when $u=0$. The data sampling can be used to detect the close return pairs, which consist of two successive points nearing each other, and indicate the existence of a periodic orbit nearby. Because of the ergodic character of the orbits on a strange attractor, we can get many such pairs if the data string is long enough. Suppose that $\boldsymbol{x}_{i,1}$ and $\boldsymbol{x}_{i,2}$ are used to denote the first point and its successive point of the ith collected return pair, $i = 1, 2, \ldots, M$, respectively, where M is the maximum number of collected return pairs. When the first close return pair has been detected (if it is within a predesignated region), taking the first point $x_{1,1}$ of this pair as a reference point, a number of close return pairs nearing the reference point can be detected,

$$|\boldsymbol{x}_{i,1} - \boldsymbol{x}_{1,1}| \leqslant \varepsilon_1, \quad |\boldsymbol{x}_{i,2} - \boldsymbol{x}_{1,1}| \leqslant \varepsilon_2, \quad i = 1, 2, \ldots, M.$$

We define the mean value as

$$\boldsymbol{x}^* = \frac{1}{2M} \sum_{i=1}^{M} [\boldsymbol{x}_{i,1} + \boldsymbol{x}_{i,2}], \tag{9}$$

where $\boldsymbol{x}^*$ is regarded as an approximate fixed point. This fixed point can be used to define a restriction condition $|\boldsymbol{x}(t) - \boldsymbol{x}^*(t)| < \varepsilon$ within which the control input signal $\boldsymbol{u} \neq 0$.

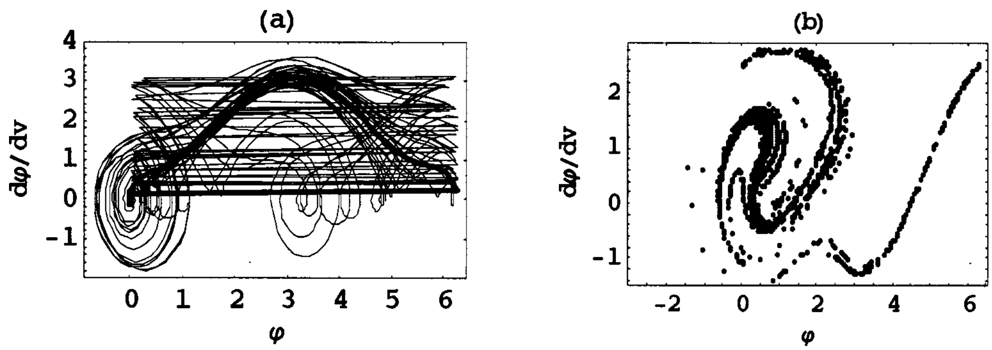

FIG. 1. Chaotic phase trajectory attractor and Poincaré maps. (a) Chaotic attractor of phase trajectory and (b) Poincaré maps.

III. NUMERICAL SIMULATIONS

A. Control of chaotic attitude motion of a spacecraft

In recent years, different approaches of chaos control are applied in the field of spacecraft techniques [21–24]. In this paper the SC method is utilized to control the chaotic attitude motion of a spacecraft, the dynamical equations of which are described in Ref. [24] as

$$\frac{d^2\varphi}{dv^2} - \frac{2e\sin v}{1+\cos v}\left(1+\frac{d\varphi}{dv}\right) + \frac{K\sin 2\varphi}{1+e\cos v} + \frac{\gamma}{(1+\cos v)^2}\frac{d\varphi}{dv} + \alpha\frac{\cos(\varphi+v+\omega)-3\cos(\varphi-v-\omega)}{1+e\cos v} + u(t) = 0, \quad (10)$$

where

$$K=\frac{3(B-A)}{2C}, \quad \alpha=\frac{I\mu_m}{2C\mu}\sin i,$$

$$\gamma=\frac{cp\sqrt{p}}{C\sqrt{\mu}},$$

and φ is the libration angle of the spacecraft in the orbital plane; v is the true anomaly of the spacecraft measured from perigee; p, e, i, and ω are semiparameter, eccentricity, angle of inclination, and the argument of perigee of the orbit, respectively; μ and μ_{m} are the gravitational and magnetic constants of the earth, respectively; A, B, and C are the principal moments of inertia of the spacecraft; and M_c is the control torque provided by the actuator. Chaotic attitude motion of the controlled spacecraft can be numerically demonstrated by integrating Eq. (10), which can be rewritten as

$$\frac{dx_1}{dv} = x_2 + u_1 = f_1(x_1,x_2) + u_1(x_1,x_2),$$

$$\begin{aligned}\frac{dx_2}{dv} &= \frac{2e\sin v}{1+e\cos v}(1+x_2) - \frac{K\sin 2x_1}{1+e\cos v} - \frac{\gamma}{(1+\cos v)^2}x_2 - \alpha\frac{\cos(x_1+v+\omega)-3\cos(x_1-v-\omega)}{1+e\cos v} + u_2 \\ &= f_2(x_1,x_2) + u_2(x_1,x_2), \quad (11)\end{aligned}$$

where $x_1=\varphi$, $x_2=d\varphi/dv$, and $u=[u_1,u_2]^{\mathrm{T}}$. The fourth-order Runge-Kutta method is used for the numerical integration. The chaos occurs when the parameters are fixed at $e=0.04$, $K=1.0$, $\gamma=0.2$, $\alpha=0.7$, and $\omega=0.1$ without input control ($u=0$). Corresponding chaotic phase trajectory attractor and Poincaré maps are depicted in Figs. 1(a) and 1(b).

We decompose the function $\boldsymbol{f}(\boldsymbol{x})$ into functions $\boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{h}(\boldsymbol{x})$ according to Eq. (2):

$$\boldsymbol{g}(x_1,x_2) = \mathbf{A}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\boldsymbol{h}(x_1,x_2) = -\boldsymbol{D}(x_2,x_1) = \begin{Bmatrix} 0.5x_1 \\ f_2(x_1,x_2)+0.5x_2 \end{Bmatrix}.$$

Then

$$(\boldsymbol{f}+\boldsymbol{D})(x_1,x_2) = \mathbf{A}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix},$$

where $\boldsymbol{f}+\boldsymbol{D}=\boldsymbol{f}-\boldsymbol{h}$ is a linear mapping and matrix $\mathbf{A}$ has negative real eigenvalues $(-0.5,-0.5)$. Obviously, the stability condition of linear system (7) is satisfied. The following control input can be used to stabilize certain desired periodic orbits embedded in the chaotic attractor of Eq. (11),

$$\begin{aligned}\boldsymbol{u}(x_1,x_2) &= \boldsymbol{D}(\boldsymbol{x}) - \boldsymbol{D}(\boldsymbol{x}^*) \\ &= \begin{Bmatrix} -0.5x_1+0.5x_1^* \\ -f_2(x_1,x_2)-0.5x_2+f_2(x_1,x_2)+0.5x_2^* \end{Bmatrix} \\ &\quad \text{if } |\boldsymbol{x}-\boldsymbol{x}^*| < \varepsilon \\ &= 0 \quad \text{otherwise}. \quad (12)\end{aligned}$$

After obtaining the characteristics of the system (10) without input signal, the SC method can be applied to control the chaotic motion of the spacecraft onto the period-1 trajectory. The numerical integration begins from the initial value $(x_1,x_2)^{\mathrm{T}}=(0,0)^{\mathrm{T}}$. A period-1 orbit $\boldsymbol{x}^*(T)$ is approximately estimated by Eq. (9) at $(5.0237\times10^{-1}$ and $7.4536\times10^{-1})^{\mathrm{T}}$ for $M=3$ and $\varepsilon_1=\varepsilon_2=5\%$.

Figures 2(a)–2(d) show the results of stabilization of the unstable period-1 orbit with $\varepsilon=0.06$. Figure 2(a) shows the process of stabilization of period-1 orbit. After a transient process, the system comes into the periodic regime corresponding to the unstable orbit $\boldsymbol{x}^*$ at 31st sampled time $v=31T$. Maintaining the control, we find that the error δ

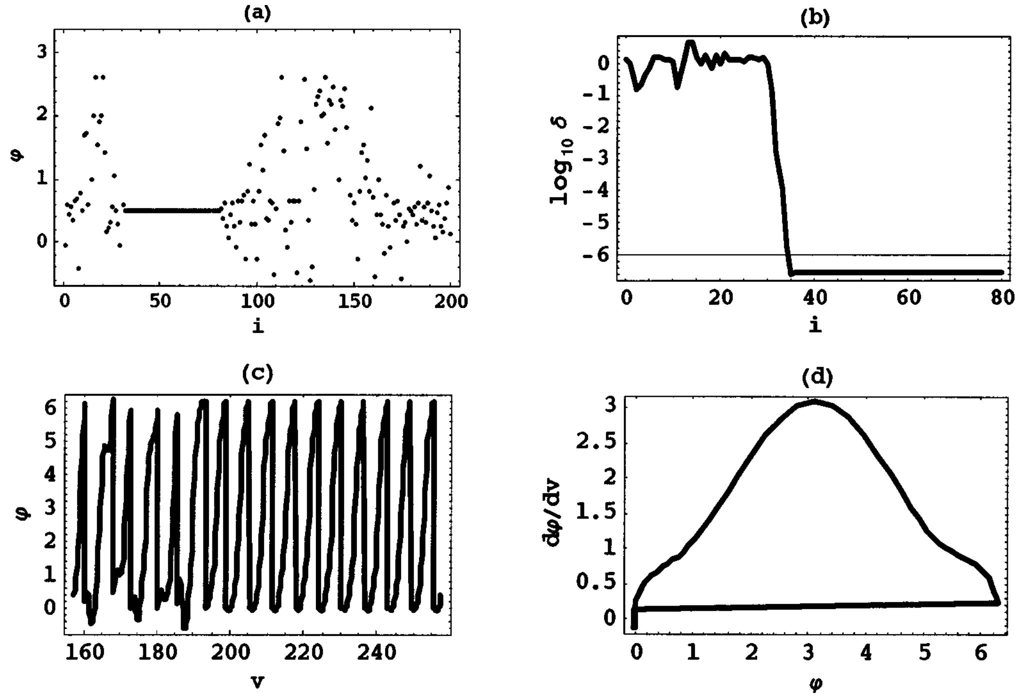

FIG. 2. Results of stabilization of the unstable period-1 orbit with $\varepsilon=0.06$. (a) Process of stabilization of period-1 orbit, (b) the plot of $\log_{10}\delta$ vs i, and (c) time history of $\varphi(\nu)$ and (d) phase trajectory of period-1.

$=|\boldsymbol{x}(iT)-\boldsymbol{x}[(i-1)T]|$ between the present sampled point and its previous point rapidly decreases with each step $i(i=1,2,\ldots,v=iT)$ and eventually achieves less than 10^{-6}. It means that the period-1 UPO is automatically detected in the control process with increasing accuracy [see Fig. 2(b)]. Figure 2(b) indicates the fast convergence property too. We maintain the control to 80th sampled time $v=80T$ and then turn it off. The time history of stabilization of the period-1 trajectory is shown in Fig. 2(c). The detected period-1 orbit is plotted in two-dimesional (2D) $(\varphi,d\varphi/dv)$ space [see Fig. 2(d)] and the period-1 UPO is embedded within the chaotic attractor shown in Fig. 1(a). The fixed point corresponding to the 2D period-1 on the sampled surface is detected at $(5.0227\times10^{-1},7.4495\times10^{-1})^T$ in the control process.

The time history of the input perturbation signal u_2 for stabilization of period-1 UPO is shown in Figs. 3(a)–3(c) when $\varepsilon=0.06$, 0.3, and 1.5. The signal u_2 is taken as zero all the time before it satisfies the condition $|\boldsymbol{x}-\boldsymbol{x}^*|<\varepsilon$. The per-

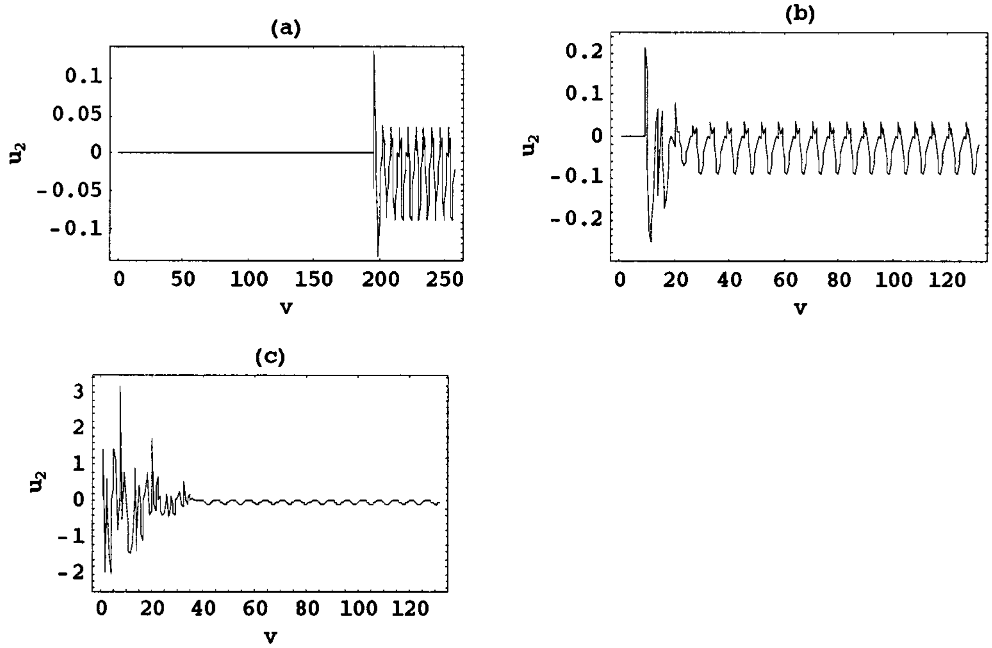

FIG. 3. Time histories of the input signal $u_2(\nu)$. (a) $\varepsilon=0.06$, (b) $\varepsilon=0.3$, and (c) $\varepsilon=1.5$.

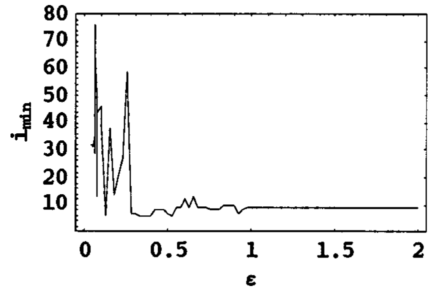

FIG. 4. Influence of restriction value ε on convergence velocity of control process, $i_{\min}$ vs ε.

turbation signal u_2 of the transient process is rather large in the case of $\varepsilon=1.5$. The u_2 is always small including the transient process; while the wait time for control on average is now longer in the case of $\varepsilon=0.06$, the control system is switched on $(u\neq 0)$only when the trajectory $\boldsymbol{x}(v)$ comes near the period-1 trajectory $\boldsymbol{x}^{*}(v)$ at certain time, namely, when the condition $\varepsilon<0.06$ is satisfied. In Fig. 4 the influence of restriction range ε on the convergence velocity of control process is illustrated. The shortest control time $v_{\min}=i_{\min}T$ is the shortest time of control process when the error $\delta=|\boldsymbol{x}(iT)-\boldsymbol{x}[(i-1)T]|$ achieves 10^{-6} and $i_{\min}$ is the shortest control steps. It is shown that the convergence of control process is fast ($i_{\min}\in[7,10]$) when the initial conditions are far from the periodic orbit, namely, when the perturbation restriction ε is taken as $\varepsilon\in[0.26,\infty)$. The control steps $i_{\min}$ fluctuate in interval [6,76] when $\varepsilon\in[0.04,0.25]$. We can choose an appropriate value of ε to suit different requirement of best control; for example, $\varepsilon=0.3$ is corresponding to a fast convergence process (see Fig. 4 and the signal u of the transient process is also small [see Fig. 3(b)]. To maintain the stabilization on the periodic orbit, the feedback of the perturbation is sufficient small.

Note a comparison between the OGY as well as OPCL method and the above-mentioned SC method. The perturbation in the OGY method (OPCL method) is applied only at the moment when the state of the system is close to the fixed point, since they use a linear approximation for the deviations from the fixed point. In the SC method the control can be started at any moment by choosing appropriate restriction condition of perturbation ε, and the chaotic behavior of the system can be interchanged easily by the perturbation input $\boldsymbol{u}(t)$ to adapt requirements stabilizing different periodic orbit. It seems that the SC method has more flexibility and convenience than the OGY method as well as OPCL method. The results of the flexible control of the chaos to unstable period-1 or period-2 orbit are shown in Figs. 5(a) and 5(b); after free running, the control is turned on at the 20th time step ($v=20T$) and the chaotic orbit is stabilized on period-1. After the maintenance of the control for 50 steps, the orbit returns to chaotic again when the control is turned off. We turn on the control again to stabilize the period-2 orbit at the 100th step, then maintain it for 60 steps.

Besides, the results of the stabilization of the period-2 orbit of the spacecraft attractor are shown in Fig. 6(a) and 6(b).

To investigate the influence of noise, we add terms $\sigma\xi_1$ and $\sigma\xi_2$ to the right-hand sides of Eq. (11), where ξ_1, and ξ_2 are two independent random functions, having mean value 0 and mean-squared value 1. Figure 7 Shows the results of stabilization of period-1 orbit for two different levels of noise with $\sigma=0.1$ and $\sigma=0.01$. There are no bursts into the region far from the UPO even for relatively large noise. The increase of noise leads to the increase of error δ and the smearing out of the period-1 orbit.

B. Control of chaotic motion of Rössler system

As another example, we consider the Rössler system described by

$$\frac{dx_1}{dt}=-x_2-x_3,\quad \frac{dx_2}{dt}=x_1+0.2x_2,$$

$$\frac{dx_3}{dt}=0.2+x_3(x_1-5.7). \tag{13}$$

We decompose the function $\boldsymbol{f}(\boldsymbol{x})$ as follows:

$$\boldsymbol{f}(\boldsymbol{x})=\begin{bmatrix}0&-1&-1\\1&-\beta&0\\0&0&-5.7\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\end{Bmatrix}+\begin{Bmatrix}0\\(0.2+\beta)x_2\\0.2+x_1x_3\end{Bmatrix}=\mathbf{A}\boldsymbol{x}+\boldsymbol{h}(\boldsymbol{x}),$$

where β is a constant, which can be selected to satisfy the stability criterion of linear system. Then the eigenvalues of the matrix $\mathbf{A}$ can be written as

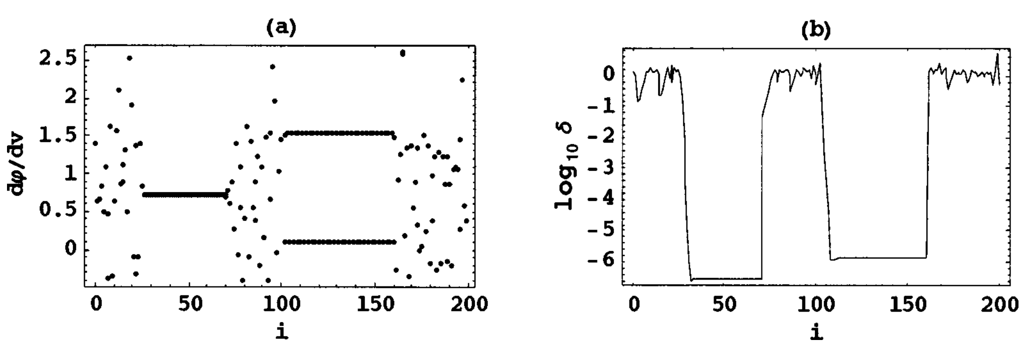

FIG. 5. The flexible control of the chaotic motion of the spacecraft system. (a) Process of stabilization of period-1 and period-2 orbits and (b) the plot of $\log_{10}\delta$ vs i.

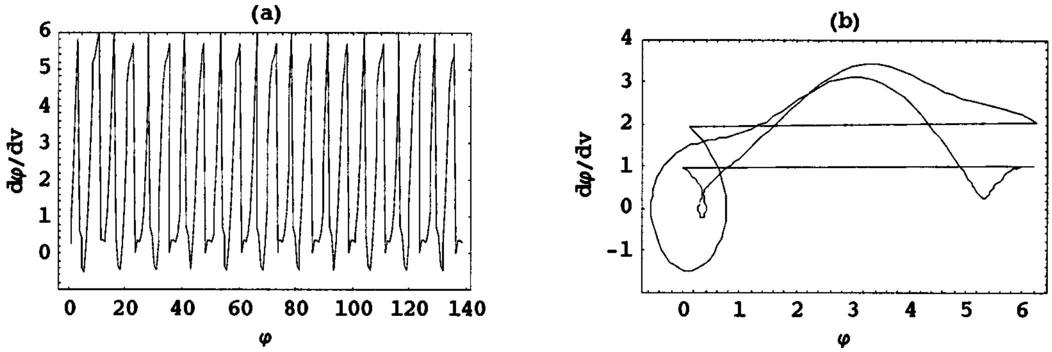

FIG. 6. Results of stabilization of the unstable period-2 orbit with $\varepsilon=0.3$. (a) Time history and (b) phase trajectory.

$$\lambda_1=-5.7, \quad \lambda_{2,3}=\frac{-\beta\pm\sqrt{\beta^2-4}}{2}.$$

Obviously all eigenvalues of matrix **A** have negative real parts only when $\beta>0$. Therefore the following control input $\boldsymbol{u}(t)$ can be used for stabilization:

$$\boldsymbol{u}(t)=\begin{Bmatrix} 0 \\ -\gamma(x_2-x_2^*) \\ -x_1x_3+x_1^*x_3^* \end{Bmatrix} \quad \text{if } |\boldsymbol{x}-\boldsymbol{x}^*|<\varepsilon$$
$$=0 \quad \text{otherwise},$$

where $\gamma=0.2+\beta$, $\beta>0$.

The results of the stabilization of the period-3 cycle of the Rössler system are illustrated in Figs. 8(a)–8(c) at $\varepsilon=2$ and $\gamma=1.2$ with period $T=17.5$. As it is expected, the perturbation becomes very small after a transient process. It is shown in Figs. 8(a)–8(c) that the control is quite simple and effective for stabilizing UPOs.

The constant γ cannot be taken very large, for example, when $\gamma>11.3$, it leads to an unsuccessful control process due to a large perturbation input $u_2(t)$. Such problem can be solved by restriction of perturbation, namely, let $u_2=U_0$ when $u_2\geqslant U_0$ and $u_2=-U_0$ when $u_2\leqslant -U_0$, where $U_0>0$ is a saturating value of the perturbation. Figs. 9(a) and 9(b) show the results of stabilization of the period-3 UPO of the Rössler attractor at $\gamma=15$, $\varepsilon=2$, and $U_0=0.08$.

C. Control of chaotic motion of two coupled Duffing oscillators

The SC method also can be applied to high-dimensional chaotic systems. As an example we consider a four-dimensional nonautonomous system consisting of two coupled Duffing oscillators [15] described by

$$\ddot{\xi}+a\dot{\xi}+\xi^3=\eta+b\cos(t),$$
$$\ddot{\eta}+c\dot{\eta}+\eta^3=\xi. \tag{14}$$

The first oscillator is driven by an external periodic force, and two oscillators interact with each other by ξ and η. When the parameters are fixed at $a=0.2$, $b=10.0$, and $c=0.45$, the chaotic behavior of the coupled Duffing oscillators can be numerically demonstrated by integrating Eq. (14), which can be rewritten as

$$\begin{aligned} \dot{x}_1&=x_2,\\ \dot{x}_2&=-ax_2-x_1^3+x_3+b\cos t,\\ \dot{x}_3&=x_4,\\ \dot{x}_4&=x_1-cx_4-x_3^3, \end{aligned} \tag{15}$$

where $x_1=\xi$, $x_2=\dot{\xi}$, $x_3=\eta$, and $x_4=\dot{\eta}$. According to Eq. (2), we suitably decompose the function $\boldsymbol{f}(\boldsymbol{x}(t))$ as

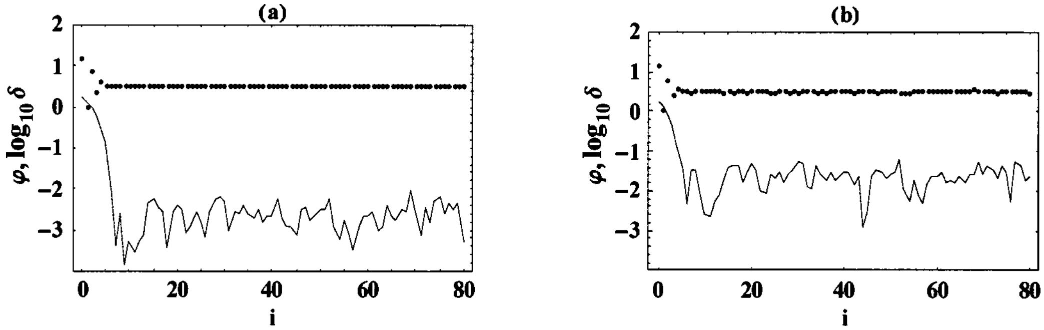

FIG. 7. The effect of noise in stabilizing a chaotic trajectory onto a period-1 orbit. (a) $\sigma=0.01$ and (b) $\sigma=0.1$

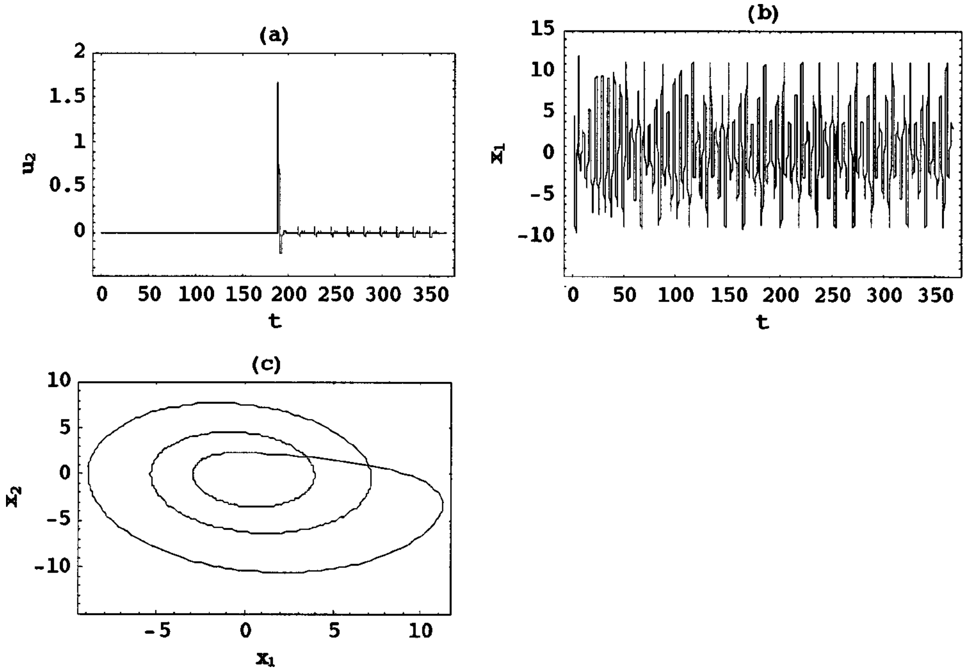

FIG. 8. Results of stabilization of the period-3 UPO of the Rössler attractor at $\gamma=1.2$, $\varepsilon=2$. (a) History of perturbation u_2, (b) history of the state variable x_1, and (c) (x_1,x_2) phase portrait of the period-3 UPO.

$$\boldsymbol{f}(t) = \mathbf{A}\boldsymbol{x}(t) - \boldsymbol{D}(\boldsymbol{x}(t)) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -a & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -c \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} - \begin{Bmatrix} -x_1 \\ x_1^3 - b\cos t \\ -x_3 \\ -x_1 + x_3^3 \end{Bmatrix}.$$

Then we obtain

$$(\boldsymbol{f}+\boldsymbol{D})(x_1,x_2,x_3,x_4) = \mathbf{A}\boldsymbol{x},$$

where matrix $\mathbf{A}$ has negative real eigenvalues $(-1,-a,-1,-c)$. This is the simplest configuration of matrix $\mathbf{A}$ whose all eigenvalues are negative real counts for high-dimensional systems. Obviously, the stability condition of the error linear system (7) is satisfied, and the following control input can be used to stabilize certain desired periodic orbit embedded in the chaotic attractor of Eq. (15),

$$\boldsymbol{u}(t) = \boldsymbol{D}(\boldsymbol{x}) - \boldsymbol{D}(\boldsymbol{x}^*) = \begin{Bmatrix} -x_1 + x_1^* \\ x_1^3 - x_1^{*3} \\ -x_3 + x_3^* \\ -x_1 + x_3^3 + x_1^* - x_3^{*3} \end{Bmatrix} \quad \text{if } |\boldsymbol{x}-\boldsymbol{x}^*| < \varepsilon$$
$$= 0 \quad \text{otherwise}.$$

Figures 10(a)–10(c) shows the chaotic attractor of the coupled Duffing oscillator in 2D subspaces (x_1,x_2), (x_3,x_4), and (x_1,x_3), respectively. The results of stabilization of the unstable period-1 orbit are shown in Figs. 10(d)–10(f).

IV. CONCLUSIONS

An important method (SC method) of chaos control based on the stability criterion of linear system is proposed in this paper. The construction of a special form of a time-

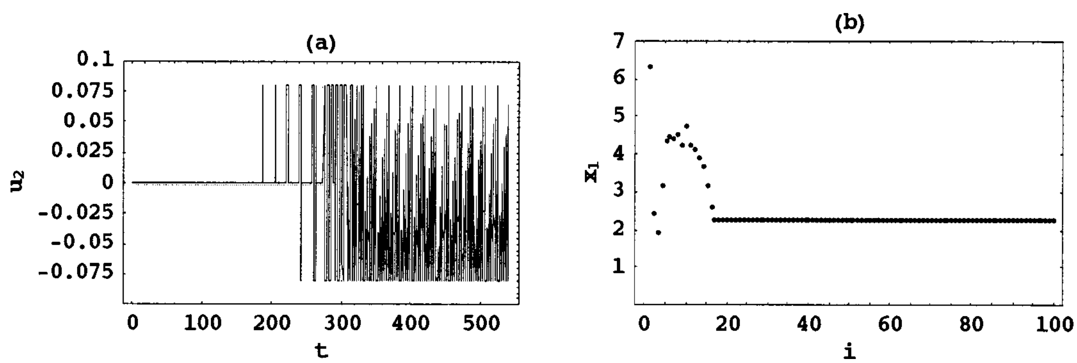

FIG. 9. Results of stabilization of the period-3 UPO of the Rössler attractor at $\gamma=15$, $\varepsilon=2$, and $U_0=0.08$. (a) History of perturbation u_2, (b) process of stabilization of the period-3 UPO with period $T=17.5$.

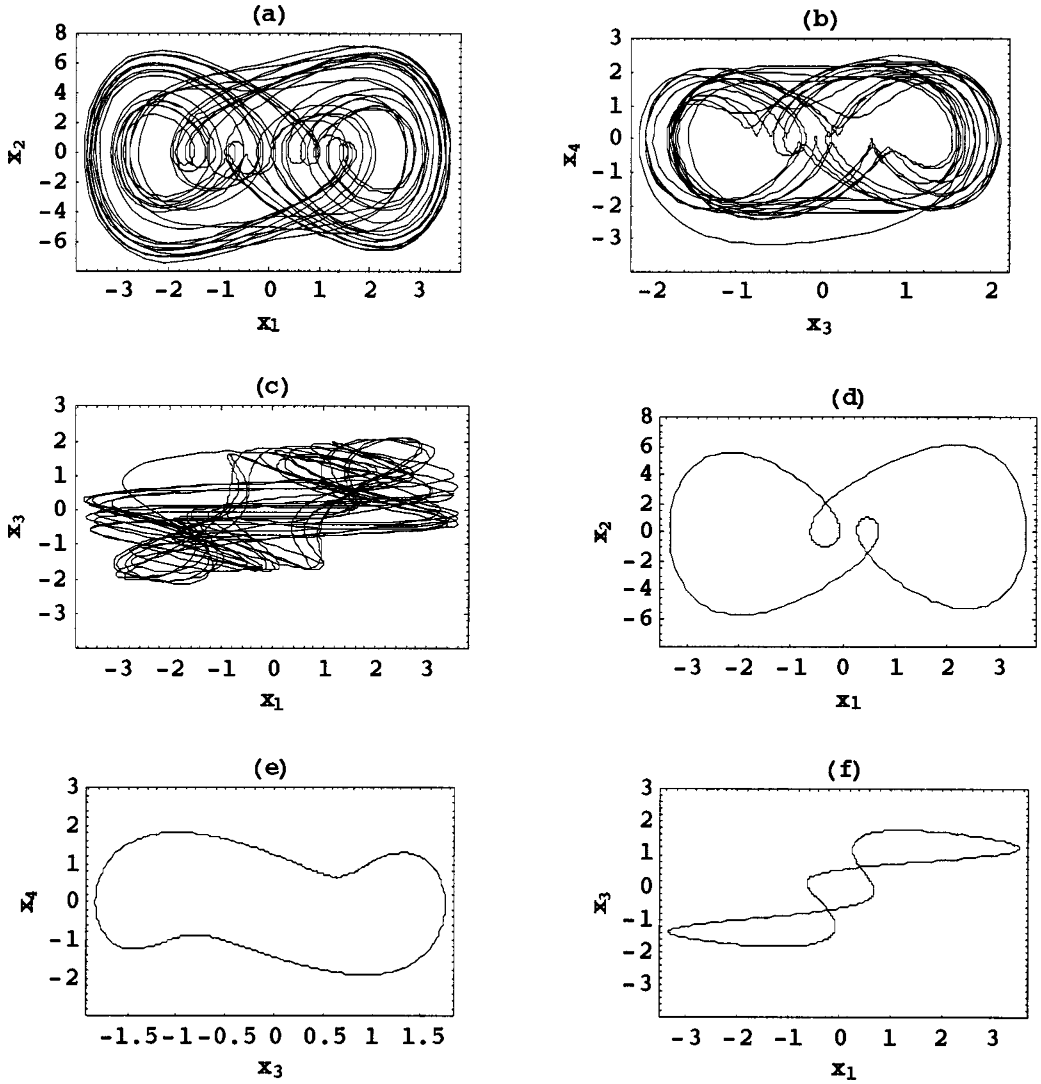

FIG. 10. The attractor of the coupled Duffing oscillator in 2D subspaces (x_1,x_2), (x_3,x_4), and (x_1,x_3), respectively. (a)–(c) The chaotic attractor and (d)–(f) results of stabilization of the unstable period-1 orbit.

continuous nonlinear perturbation feedback in the SC method does not change the form of the desired UPO for chaos control. The close return pair technique is utilized to estimate a desired periodic orbit chosen from numerous UPOs embedded within a chaotic attractor. This method does not require linearization of the system around the stabilized orbit and estimation of the derivative at UPOs. The calculation of the maximal Lyapunov exponent of the UPOs analyzing the local stability of the system and selecting the range of control parameter is not needed. It is unnecessary to start the control at the moment when the state of system is close to the desired periodic orbit. The control can be started at any moment by choosing appropriate perturbation restriction condition ε, and therefore the chaotic behavior of the system can be changed to any desired orbit easily by the perturbation input $\boldsymbol{u}(t)$. The validity of stabilization is shown by numerical simulation even for high-dimensional systems. It seems that more flexibility and convenience are main advantages of this method.

The complexity of the experimental realization of the SC method is mainly the input of desired UPOs. The method also relies on explicit knowledge of the system dynamics. We will solve these problems using delayed feedback input signal in another paper. Moreover this method can be applied for some possible goal behavior except UPOs embedded in chaotic attractor. Detailed discussion can be given also in another paper.

ACKNOWLEDGMENT

This research was supported by the National Natural Science Foundation of China (Project No.10272074).

[1] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[2] L. M. Pecora and T. L. Carrol, Phys. Rev. Lett. **64**, 821 (1990).
[3] H. J. Yu, and Y. Z. Liu, Phys. Lett. A **314**, 292 (2003).
[4] W. L. Ditto, S. N. Rauseo, and M. L. Spano, Phys. Rev. Lett. **65**, 3211 (1990).
[5] E. R. Hunt, Phys. Rev. Lett. **68**, 1259 (1992).
[6] J. Starrett and R. Tagg, Phys. Rev. Lett. **74**, 1974 (1995).
[7] A. Garfinkel, M. Spano, W. L. Ditto, and J. Weiss, Science **257**, 1230 (1992).
[8] T. Shinbrot, C. Grebogi, E. Ott and J. A. Yorke, Nature (London) **363**, 411 (1993).
[9] F. J. Romeiras, C. Grebogi, E. Ott, and W. P. Dayawansa, Physica D **58**, 165 (1992).
[10] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[11] V. Petrov, V. Gáspár, J. Masere, K. Showalter, Nature (London) **361**, 240 (1993).
[12] E. Atlee Jackson and I. Grosu, Physica D **85**, 1 (1995).
[13] H. Sabbagh, Chaos, Solitons Fractals **11**, 1213 (2000).
[14] S. R. Bishop and D. Xu, Phys. Rev. E **54**, 3204 (1996).
[15] D. Xu and S. R. Bishop, Phys. Lett. A **210**, 273 (1996).
[16] L. Q. Chen and Y. Z. Liu, Phys. Lett. A **262**, 350 (1999).
[17] B. Hübinger, R. Doerner, W. Martienssen, M. Herdering, R. Pitka, and U. Dressler, Phys. Rev. E **50**, 932 (1994).
[18] M. A. Matias and J. Guemez, Phys. Rev. Lett. **72**, 1455 (1994).
[19] B. Hübinger, R. Doemer, and W. Martienssen, Z. Phys. B: Condens. Matter **90**, 103 (1993).
[20] T. Ushio, Phys. Lett. A **198**, 14 (1995).
[21] A. P.M. Tsui and A. J. Jones, Physica D **135**, 41 (2000).
[22] Z. M. Ge *et al.*, J. Sound Vib. **217**, 807 (1998).
[23] L. Q. Chen and Y. Z. Liu, Int. J. Non-Linear Mech. **37**, 493 (2002).
[24] L. Q. Chen and Y. Z. Liu, Acta Mech. Sin. **19**, 71 (2003).